$$I_\Gamma = 2L. \quad (7)$$

Equation (7) shows that between the L-integral and the Bueckner work conjugate integral there is a simple but inherent relationship. We need not to know the obvious function expressions of complex potentials for the crack beforehand, but the traction-free conditions must be satisfied.

If there are two displacement-stress fields, namely, u_i, σ_{ij} and $u_i^{(II)}$, $\sigma_{ij}^{(II)}$, using the Betti's reciprocal theorem to the region bounded by crack borders, one can divide the contour C into C_L, C_R and C_+, C_-, where C_L, C_R are circles around the left and the right crack tips and C_+, C_- are the straight line along the upper and lower crack faces, respectively. Because the stresses are free on the crack faces, then $I_\Gamma = I_{CL} + I_{CR}$. In this case, if integrals I_{CL} and I_{CR} can be evaluated for some displacement-stress fields, then the path-independent integral I_Γ can also be defined and evaluated. If the displacement-stress fields are defined by Eqs. (3) and (4), we can deduce the following relationship between the L-integral and the stress intensity factors:

$$L = -\frac{3(\kappa-1)a}{4\mu}(K_{1L}K_{2L} + K_{1R}K_{2R}) \quad (8)$$

where $K_L = K_{1L} + iK_{2L}$, $K_R = K_{1R} + iK_{2R}$ are stress intensity factors at the left and the right crack tips, respectively. κ and μ are elastic constants.

3 Discussion

The complex potentials of the center crack, $\varphi_1(z), \omega_1(z), \varphi_2(z), \omega_2(z)$, were obtained by Chen and Shi [3] by using the same method obtained the eigenfunction expansion form by Rice [4] in interfacial cracks for dissimilar material. The stress and displacement fields that are obtained from these complex potentials satisfy the traction-free conditions on the crack faces and the continuous condition along the entire interface.

A supplemental displacement-stress field defined by the complex potentials $\varphi_1^{(II)}(z), \omega_1^{(II)}(z), \varphi_2^{(II)}(z), \omega_2^{(II)}$ is introduced. The relations between $\varphi_1^{(II)}(z), \omega_1^{(II)}(z), \varphi_2^{(II)}(z), \omega_2^{(II)}$ and $\varphi_1(z), \omega_1(z), \varphi_2(z), \omega_2(z)$ are analogous to Eq. (4).

In a similar manner, the displacement and stress of the (II) field are presented in Eq. (5). They satisfy the traction-free conditions on the crack faces also. The corresponding displacement and stress components will be substituted into Eq. (6). Note that the curve Γ can be divided into two sections: curve Γ_1 of the upper plane and the curve Γ_2 of the below plane. The deductions of Eqs. (6) to (7) relate to the equilibrium equations in plane problems and traction-free conditions only, but don't involve the material parameter. The process is the same as the above homogeneous isotropic material. Finally, we still obtain Eq. (7) in the interface crack, that is $I_\Gamma = 2L$.

Between L-integral and stress intensity factors there is the following relation:

$$L = -\left(\frac{\kappa_1 - 1}{\mu_1} + \frac{\kappa_2 - 1}{\mu_2}\right)\frac{3(K_{1L}K_{2L} + K_{1R}K_{2R})}{8\cosh^2(\pi\varepsilon)}a \quad (9)$$

where ε is "oscillation index" and $K_L = K_{1L} + iK_{2L}$, $K_R = K_{1R} + iK_{2R}$ are complex stress intensity factors at the left and the right crack-tips, respectively. They cannot be separated into the pure I model and II model; κ_1, μ_1 and κ_2, μ_2 stand for the material parameters of the upper and lower plane.

For anisotropy material, the Lekhenitski complex potential theory needs to be used ([5]). According to the need of the Bueckner work conjugate integral, the subsidiary stress-displacement fields, which represents (II) field, are

$$\varphi^{(II)}(z_1) = -iz_1\varphi'(z_1)$$

$$\psi^{(II)}(z_2) = -iz_2\psi'(z_2) + 2i\bar{z}_2\varphi'(z_1). \quad (10)$$

The stress fields caused by Eq. (10) satisfy the traction-free conditions. The relation between the (II) field and a physical stress field are analogous to Eq. (5).

By proceeding in the same manner as the isotropic case from Eq. (6) to Eq. (7), we draw a conclusion $I_\Gamma = 2L$.

It can be seen that a simple but inherent relation between the L-integral and the Bueckner work conjugate integral is right all along, although the characteristic of material is more complex than isotropic and the complex potentials in these two cases are more different with in isotropic.

4 Conclusions

Using the Bueckner work conjugate integral through introducing a special subsidiary stress-displacement field, one can render the L-integral. The relation between L-integral and the Bueckner work conjugate integral seems independent of the stress oscillatory singularities on the interface crack tips and the eigenroot in the anisotropy. It is found that the L-integral, from the mathematical point of view as well as in principle, is arising from the Betti's reciprocal theorem. This means that the Bueckner work conjugate integral is a more general path-independent integral than others are. Using the Bueckner integral through choosing a different subsidiary stress-displacement field could render any other path-independent integrals.

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A Note on the Driving Traction Acting on a Propagating Interface: Adiabatic and Non-Adiabatic Processes of a Continuum

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An expression for the driving traction on an interface is derived for an arbitrary continuum undergoing an arbitrary thermomechanical process which may or may not be adiabatic.

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1 Summary

In this note we derive an expression for the "driving traction," or Eshelby force ([1]), acting on a propagating interface in a continuum. The interfaces that we have in mind might represent, for example, a shock wave or a boundary between two phases of a material, and the thermomechanical processes which the continuum is permitted to undergo may or may not be adiabatic. From the perspective of irreversible thermodynamics, the driving traction corresponds to a "thermodynamic affinity"; see, for example, [2–4]. It plays a central role in modeling the kinetics of phase transformations by characterizing the rate of propagation of phase boundaries (e.g., see [5–8]).

The derivation sketched below makes no assumptions about the constitutive law for the continuum under consideration. When specialized to a thermoelastic material, the expression for the driving traction obtained here has certain similarities with the Legendre transform of the Helmholtz free-energy $\hat{\psi}(\mathbf{F},\theta)$ with respect to *both* the deformation gradient tensor $\mathbf{F}$ and the temperature θ, as well as with the Legendre transform of the internal energy $\hat{\varepsilon}(\mathbf{F},\eta)$ with respect to $\mathbf{F}$ and the specific entropy η.

The result derived here generalizes an earlier one which had been established for non-adiabatic processes ([9,10,5]). This former characterization of driving traction was not valid in adiabatic processes, and therefore did not, in particular, apply to shock waves in classical gas dynamics or to impact-induced rapidly moving phase boundaries in solids. A one-dimensional version of the present result was obtained in [11].

2 Momentum and Energy

Consider a body which occupies a region R in a reference configuration. Let $\mathbf{x}\in R$ denote the position of a particle in this configuration and let t denote time. Consider a thermomechanical process of this body on some time interval $[t_1,t_2]$ which is characterized by the motion $\mathbf{y}(\mathbf{x},t)$, body force per unit mass $\mathbf{b}(\mathbf{x},t)$, Piola-Kirchhoff stress $\boldsymbol{\sigma}(\mathbf{x},t)$, heat flux $\mathbf{q}(\mathbf{x},t)$, heat supply $r(\mathbf{x},t)$ and internal energy per unit mass $\varepsilon(\mathbf{x},t)$. Suppose that during this process $\mathbf{y}$ is continuous with piecewise continuous first and second derivatives on $R\times[t_1,t_2]$; $\mathbf{b}(\cdot,t)$ and $r(\cdot,t)$ are continuous on R; $\boldsymbol{\sigma}(\cdot,t)$ and $\mathbf{q}(\cdot,t)$ are piecewise continuous with piecewise continuous gradient on R; and ε is piecewise continuous with piecewise continuous first derivatives on $R\times[t_1,t_2]$. During this process, the usual balance laws of linear and angular momentum and the first law of thermodynamics require that for any subregion D,

$$\int_{\partial D}\boldsymbol{\sigma}\mathbf{n}\,dA+\int_D\rho\mathbf{b}\,dV=\frac{d}{dt}\int_D\rho\mathbf{v}\,dV, \tag{1}$$

$$\int_{\partial D}\mathbf{y}\times\boldsymbol{\sigma}\mathbf{n}\,dA+\int_D\mathbf{y}\times\rho\mathbf{b}\,dV=\frac{d}{dt}\int_D\mathbf{y}\times\rho\mathbf{v}\,dV, \tag{2}$$

$$\int_{\partial D}(\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{v}+\mathbf{q}\cdot\mathbf{n})dA+\int_D(\rho\mathbf{b}\cdot\mathbf{v}+\rho r)dV=\frac{d}{dt}\int_D\left(\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}+\rho\varepsilon\right)dV. \tag{3}$$

Here $\mathbf{v}=\dot{\mathbf{y}}$ denotes particle velocity, $\rho(\mathbf{x})$ is the mass density in the reference configuration which is assumed to be continuous on R, and $\mathbf{n}$ is a unit outward normal vector on ∂D.

At a point in R at which the fields are smooth the balance laws (1)–(3) yield the usual field equations

$$\mathrm{Div}\,\boldsymbol{\sigma}+\rho\mathbf{b}=\rho\dot{\mathbf{v}}, \tag{4}$$

$$\boldsymbol{\sigma}\mathbf{F}^T=\mathbf{F}\boldsymbol{\sigma}^T, \tag{5}$$

$$\boldsymbol{\sigma}\cdot\dot{\mathbf{F}}+\mathrm{Div}\,\mathbf{q}+\rho r=\rho\dot{\varepsilon}, \tag{6}$$

where $\mathbf{F}=\mathrm{Grad}\,\mathbf{y}$ is the deformation gradient tensor.

Next, suppose that there is a surface S_t in R such that the fields $\mathbf{F}$, $\mathbf{v}$, $\mathbf{q}$, $\boldsymbol{\sigma}$ and ε suffer jump discontinuities across S_t while being continuous on either side of it. Such a surface may represent, for example, the Lagrangian image of a shock wave or an interface separating two material phases. Let $V_n\geq 0$ denote the normal velocity of propagation of this interface. We refer to the side into which V_n points as the positive side of S_t. For any field quantity $g(\mathbf{x},t)$ let $\overset{+}{g}$ and $\overline{g}$ denote the limiting values of g as a point on S_t is approached from its positive and negative side, respectively. Then, we let $[[g]]$ and $\langle g\rangle$ denote the jump and the average values of g on S_t:

$$[[g]]=\overset{+}{g}-\overline{g},\quad \langle g\rangle=\frac{1}{2}(\overset{+}{g}+\overline{g}). \tag{7}$$

At a point on S_t, the balance laws (1)–(3) yield the usual jump conditions

$$[[\boldsymbol{\sigma}\mathbf{n}]]+[[\rho\mathbf{v}]]V_n=0, \tag{8}$$

$$[[\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{v}]]+\left[\left[\rho\varepsilon+\frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\right]\right]V_n+[[\mathbf{q}\cdot\mathbf{n}]]=0. \tag{9}$$

The energy jump condition (9) can be written in the following alternative form by making use of (8) and $[[\mathbf{v}]]+V_n[[\mathbf{Fn}]]=\mathbf{0}$ which follows from the continuity of the deformation (for algebraic details see, for example, [5]):

$$([[\rho\varepsilon]]-\langle\boldsymbol{\sigma}\rangle\cdot[[\mathbf{F}]])V_n=-[[\mathbf{q}\cdot\mathbf{n}]]. \tag{10}$$

3 Rate of Entropy Production

In order to address the second law of thermodynamics one must consider two additional fields, viz. the temperature $\theta(\mathbf{x},t)$ and the entropy per unit mass $\eta(\mathbf{x},t)$. Suppose that $\theta(\cdot,t)$ is piecewise continuous with a piecewise continuous gradient on R, and that η is piecewise continuous with piecewise continuous first derivatives on $R\times[t_1,t_2]$; θ and η are permitted to suffer jump discontinuities across S_t. The rate of entropy production associated with a subregion D is defined by

$$\Gamma=\frac{d}{dt}\int_D\rho\eta\,dV-\int_{\partial D}\frac{\mathbf{q}\cdot\mathbf{n}}{\theta}dA-\int_D\frac{\rho r}{\theta}dV, \tag{11}$$

and the second law of thermodynamics requires that $\Gamma\geq 0$ for all regions D and all processes. When the region D intersects the interface S_t one can rewrite (11) in the form

$$\Gamma=\int_D\left\{\rho\dot{\eta}-\mathrm{Div}\left(\frac{\mathbf{q}}{\theta}\right)-\frac{\rho r}{\theta}\right\}dV-\int_{S_t\cap D}\left\{[[\rho\eta]]V_n+\left[\left[\frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\right]\right]\right\}dA \tag{12}$$

by carrying out a standard calculation; e.g. see page 116 of [12]. The first term in (12) represents the entropy production rate in the bulk of the body and the second term is associated with the moving interface. Let Γ_s denote the rate of entropy production due to the propagating surface:

$$\Gamma_s=-\int_{S_t}\left\{[[\rho\eta]]V_n+\left[\left[\frac{\mathbf{q}\cdot\mathbf{n}}{\theta}\right]\right]\right\}dA. \tag{13}$$

One finds by using (13) and (10), that Γ_s can be alternatively expressed as

$$\Gamma_s=\int_{S_t}\left\{\left[\left\langle\frac{1}{\theta}\right\rangle([[\rho\varepsilon]]-\langle\boldsymbol{\sigma}\rangle\cdot[[\mathbf{F}]])-[[\rho\eta]]\right]V_n+\langle\mathbf{q}\cdot\mathbf{n}\rangle\left[\left[\frac{1}{\theta}\right]\right]\right\}dA. \tag{14}$$

In an adiabatic process there is no heat transfer: $\mathbf{q}=\mathbf{0}$ and $r=0$. On the other hand if the process is not adiabatic, the typical heat conduction law, whatever it may be, involves Grad θ and therefore the partial differential equations resulting from using the constitutive relationships in the energy Eq. (6) involve (at least) the second spatial derivative of θ; thus, one usually requires the temperature to be continuous in non-adiabatic processes: $[[\theta]]=0$ on S_t. Thus in both the adiabatic and non-adiabatic cases one has $[[\theta]]\overset{\pm}{\mathbf{q}}=\mathbf{0}$ on S_t and therefore necessarily

$$\langle\mathbf{q}\cdot\mathbf{n}\rangle\left[\left[\frac{1}{\theta}\right]\right]=0 \quad \text{and} \quad \left(\left\langle\frac{1}{\theta}\right\rangle-\frac{1}{\langle\theta\rangle}\right)[[\mathbf{q}\cdot\mathbf{n}]]=0. \tag{15}$$

In view of this and (10), we can write Γ_s as

$$\Gamma_s=\int_{S_t}\frac{[[\rho\varepsilon]]-\langle\boldsymbol{\sigma}\rangle\cdot[[\mathbf{F}]]-\langle\theta\rangle[[\rho\eta]]}{\langle\theta\rangle}V_n\,dA, \tag{16}$$

or in terms of the Helmholtz free-energy $\psi=\varepsilon-\eta\theta$ as

$$\Gamma_s=\int_{S_t}\frac{[[\rho\psi]]-\langle\boldsymbol{\sigma}\rangle\cdot[[\mathbf{F}]]+\langle\rho\eta\rangle[[\theta]]}{\langle\theta\rangle}V_n\,dA. \tag{17}$$

4 Driving Traction

The rate of entropy production due to the propagating interface can be written as

$$\Gamma_s=\int_{S_t}\frac{fV_n}{\langle\theta\rangle}dA \tag{18}$$

where

$$\begin{aligned}f&=[[\rho\psi]]-\langle\boldsymbol{\sigma}\rangle\cdot[[\mathbf{F}]]+\langle\rho\eta\rangle[[\theta]]\\&=[[\rho\varepsilon]]-\langle\boldsymbol{\sigma}\rangle\cdot[[\mathbf{F}]]-\langle\theta\rangle[[\rho\eta]]\end{aligned} \tag{19}$$

is called the driving traction or Eshelby force. The second law of thermodynamics requires that $fV_n\geq 0$ on S_t which specifies the direction in which the interface is permitted to move. This result is valid for any continuum undergoing an arbitrary thermomechanical process which may or may not be adiabatic. If the process is adiabatic, (19) and (10) yield $f=-\langle\theta\rangle[[\rho\eta]]$. If it is not adiabatic, (19) specializes to $f=[[\rho\psi]]-\langle\boldsymbol{\sigma}\rangle\cdot[[\mathbf{F}]]$.

In the special case of a thermoelastic material one has $\psi=\hat{\psi}(\mathbf{F},\theta)$ and the stress and entropy are given by the constitutive relationships $\boldsymbol{\sigma}=\rho\hat{\psi}_{\mathbf{F}}$, $\eta=-\hat{\psi}_\theta$. Equivalently one has $\varepsilon=\hat{\varepsilon}(\mathbf{F},\eta)$ with the stress and temperature given by $\boldsymbol{\sigma}=\rho\hat{\varepsilon}_{\mathbf{F}}$, $\theta=\hat{\varepsilon}_\eta$. Thus for a thermoelastic material (19) can be written as

$$\begin{aligned}f&=[[\rho\hat{\psi}]]-\langle\rho\hat{\psi}_{\mathbf{F}}\rangle\cdot[[\mathbf{F}]]-\langle\rho\hat{\psi}_\theta\rangle[[\theta]]\\&=[[\rho\hat{\varepsilon}]]-\langle\rho\hat{\varepsilon}_{\mathbf{F}}\rangle\cdot[[\mathbf{F}]]-\langle\rho\hat{\varepsilon}_\eta\rangle[[\eta]]\end{aligned} \tag{20}$$

which is reminiscent of the Legendre transforms of $\rho\hat{\psi}(\mathbf{F},\theta)$ and $\rho\hat{\varepsilon}(\mathbf{F},\eta)$.

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Characterizing Damping and Restitution in Compliant Impacts via Modified K-V and Higher-Order Linear Viscoelastic Models

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1 Introduction

Time-domain models for compliant impacts have been widely used to model collision dynamics as finite-time events. The most common way to account for energy dissipation in the compliant impact model has been via the standard Kelvin-Voigt (K-V) viscoelastic model

$$F(t)=kx+c\dot{x} \tag{1}$$

in which the resulting equation of motion assumes the familiar linear form

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2x=0 \tag{2}$$

from vibration theory where $\omega_n=\sqrt{k/m}$ and $\zeta=c/(2\sqrt{km})$. The initial conditions $x(0)=0$ and $\dot{x}(0)=v_0$ yield the solution

$$x(t)=\frac{v_0}{\omega_d}\exp(-\zeta\omega_nt)\sin\omega_dt \tag{3}$$

where $\omega_d=\omega_n\sqrt{1-\zeta^2}$. If the impact duration is assumed to be a half-period of vibration associated with the damped frequency, then the exact restitution coefficient is obtained easily in terms of the dimensionless damping ratio as

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